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Application of Aboodh Transform for Solving Linear Volterra Integro-Differential Equations of Second Kind

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Abstract: In this paper, Aboodh transform is used for solving linear Volterra integro-differential equations of second kind. The technique is described and illustrated with some numerical applications. The results assert that this scheme give the exact results using very less computational work.

Keywords: Linear Volterra integro-differential equation, Aboodh transform, Convolution theorem, Inverse Aboodh transform.

1. INTRODUCTION

Functional differential equations, delay differential equations, partial differential equations, integral equations, integro-differential equations, stochastic equations etc. are the resultant of mathematical modeling of the real life problems. There are a number of process and phenomenon in different areas of science and engineering where integro-differential equations plays vital role (like in nuclear reactors, circuit analysis, wave propagation, glass forming process, nano-hydrodynamics, visco-elasticity, biological population etc.).

Volterra, during his working on population growth model, have studied the hereditary influences where both the operators (differential and integral) were seen to be appearing together in the same equation. This new equation was later on known as Volterra integro-differential equation and given in the form [1-5]

$$u^n(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt \ldots\ldots\ldots\ldots (1)$$

where $u^n(x)$ is the n[th] derivative of unknown function $u(x)$ with respect to x . Since equation (1) contains both the operator (differential and integral) so for finding the particular solution $u(x)$ of equation(1), it is very necessary to define initial conditions $u(0), u'(0), \ldots\ldots.., u^{(n-1)}(0)$.

Volterra integro-differential equation contains one or more of the derivatives of unknown function$u(x)$ with respect to x i.e. $u'(x), u''(x), u'''(x), \ldots\ldots..$ outside the integral sign. Generally Volterra integro-differential equations may be appear when an initial value problem converted into an integral equation using Leibnitz rule.

The Aboodh transform of the function $F(t)$ is defined as [6, 8, 9,]:

$$A\{F(t)\} = \frac{1}{v}\int_0^\infty F(t)e^{-vt}dt$$

$$= K(v), t \geq 0, 0 < k_1 \leq v \leq k_2$$

where A is Aboodh transform operator.

The Aboodh transform of the function $F(t)$ for $t \geq 0$ exist if $F(t)$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Aboodh transform of the function $F(t)$.

Aboodh [7] gave the application of new transform "Aboodh Transform" to partial differential equations. Aboodh et al. [8] discussed the connection of Aboodh transform with some famous integral transforms. Aboodh et al. [9] solved delay differential equations using Aboodh transformation method. Aboodh et al. [10] gave the solution of ordinary differential equation with variable coefficients using Aboodh transform. Solution of partial integro-differential equations by using Aboodh and double Aboodh transforms methods was given by Aboodh et al [11].

The aim of this work is to establish exact solutions for linear Volterra integro-differential equations of second kind using Aboodh transform without large computational work.

2. LINEARITY PROPERTY OF ABOODH TRANSFORMS [8]

$$A\{aF(t)+bG(t)\}=aA\{F(t)\}+bA\{G(t)\}$$

where a and b are arbitrary constants.

3. ABOODH TRANSFORM OF SOME ELEMENTARY FUNCTIONS [6, 9]:

S.N.	$F(t)$	$A\{F(t)\}=K(v)$
1.	1	$\frac{1}{v^2}$
2.	t	$\frac{1}{v^3}$
3.	t^2	$\frac{2!}{v^4}$
4.	$t^n, n \geq 0$	$\frac{n!}{v^{n+2}}$
5.	e^{at}	$\frac{1}{v^2-av}$
6.	$sinat$	$\frac{a}{v(v^2+a^2)}$
7.	$cosat$	$\frac{1}{v^2+a^2}$
8.	$sinhat$	$\frac{a}{v(v^2-a^2)}$
9.	$coshat$	$\frac{1}{v^2-a^2}$

4. ABOODH TRANSFORM OF THE DERIVATIVES OF THE FUNCTION $F(t)$ [6, 8]:

If $A\{F(t)\}=K(v)$ then

a) $A\{F'(t)\}=vK(v)-\frac{F(0)}{v}$

b) $A\{F''(t)\}=v^2K(v)-\frac{F'(0)}{v}-F(0)$

c) $A\{F^{(n)}(t)\}=v^nK(v)-\frac{F(0)}{v^{2-n}}-\frac{F'(0)}{v^{3-n}}-\cdots\ldots-\frac{F^{(n-1)}(0)}{v}$

5. CONVOLUTION OF TWO FUNCTIONS [12]:

Convolution of two functions $F(t)$ and $G(t)$ is denoted by $F(t)*G(t)$ and it is defined by

$$F(t)*G(t)=F*G=\int_0^t F(x)G(t-x)dx$$
$$=\int_0^t F(t-x)G(x)dx$$

6. CONVOLUTION THEOREM FOR ABOODH TRANSFORMS [8]:

If $A\{F(t)\}=H(v)$ and $A\{G(t)\}=I(v)$ then

$$A\{F(t)*G(t)\}=vA\{F(t)\}A\{G(t)\}=vH(v)I(v)$$

7. INVERSE ABOODH TRANSFORM:

If $A\{F(t)\}=H(v)$ then $F(t)$ is called the inverse Aboodh transform of $H(v)$ and mathematically it is defined as

$$F(t)=A^{-1}\{H(v)\}$$

where A^{-1} is the inverse Aboodh transform operator.

8. INVERSE ABOODH TRANSFORM OF SOME ELEMENTARY FUNCTIONS:

S.N.	$K(v)$	$F(t)=A^{-1}\{K(v)\}$
1.	$\frac{1}{v^2}$	1
2.	$\frac{1}{v^3}$	t
3.	$\frac{1}{v^4}$	$\frac{t^2}{2!}$
4.	$\frac{1}{v^{n+2}}, n \geq 0$	$\frac{t^n}{n!}$
5.	$\frac{1}{v^2-av}$	e^{at}
6.	$\frac{1}{v(v^2+a^2)}$	$\frac{sinat}{a}$
7.	$\frac{1}{v^2+a^2}$	$cosat$
8.	$\frac{1}{v(v^2-a^2)}$	$\frac{sinhat}{a}$
9.	$\frac{1}{v^2-a^2}$	$coshat$

9. ABOODH TRANSFORMS FOR LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF SECOND KIND:

In this section, we present Aboodh transform for solving linear Volterra integro-differential equations of second kind given by (1). In this work, we will assume that the kernel $k(x,t)$ of (1) is a difference kernel that can be expressed by difference $(x-t)$. The linear Volterra integro-differential equation of second kind (1) can thus be expressed as

$$\left.\begin{array}{c} u^{n}(x) = f(x) + \lambda \int_0^x k(x-t)u(t)dt \\ with\, u(0) = a_0, u'(0) = a_1, \dots, u^{(n-1)}(0) = a_{n-1} \end{array}\right\} \dots .(3)$$

Applying the Aboodh transform to both sides of(3), we have

$$v^n A\{u(x)\} = \frac{a_0}{v^{2-n}} + \frac{a_1}{v^{3-n}} + \cdots \dots + \frac{a_{n-1}}{v}$$

$$+A\{f(x)\} + \lambda A\{\int_0^x k(x-t)u(t)dt\} \dots ..(4)$$

Using convolution theorem of Aboodh transform, we have

$$A\{u(x)\} = \frac{a_0}{v^2} + \frac{a_1}{v^3} + \cdots \dots + \frac{a_{n-1}}{v^{n+1}} + \frac{1}{v^n} A\{f(x)\} + \frac{\lambda}{v^{n-1}} A\{k(x)\}A\{u(x)\} \dots \dots .(5)$$

Operating inverse Aboodh transform on both sides of(5), we have

$$u(x) = a_0 + a_1 x + \cdots \dots . + a_{n-1} \frac{x^{n-1}}{n-1!} + A^{-1}\left\{\frac{1}{v^n} A\{f(x)\}\right\} + \lambda A^{-1}\left\{\frac{1}{v^{n-1}} A\{k(x)\}A\{u(x)\}\right\} \dots ..(6)$$

which is the required solution of (3).

10. APPLICATIONS

In this section, some applications are given in order to demonstrate the effectiveness of Aboodh transform for solving linear Volterra integro-differential equation of second kind.

Application:1 Consider linear Volterra integro-differential equation of second kind

$$\left.\begin{array}{c} u'(x) = 2 + \int_0^x u(t)dt \\ with\, u(0) = 2 \end{array}\right\} \dots \dots (7)$$

Applying the Aboodh transform to both sides of (7) and using initial condition, we have

$$vA\{u(x)\} = \frac{2}{v} + \frac{2}{v^2} + A\left\{\int_0^x u(t)\, dt\right\} \dots .(8)$$

Using convolution theorem of Aboodh transform on (8) and simplify, we have

$$A\{u(x)\} = \frac{2}{v^2 - v} \dots \dots .(9)$$

Operating inverse Aboodh transform on both sides of(9), we have

$$u(x) = A^{-1}\left\{\frac{2}{v^2 - v}\right\} = 2A^{-1}\left\{\frac{1}{v^2 - v}\right\} = 2e^x \dots \dots (10)$$

which is the required exact solution of (7).

Application:2 Consider linear Volterra integro-differential equation of second kind

$$\left.\begin{array}{c} u''(x) = 1 + \int_0^x (x-t)u(t)dt \\ with\, u(0) = 1, u'(0) = 0 \end{array}\right\} \dots \dots (11)$$

Applying the Aboodh transform to both sides of (11) and using initial condition, we have

$$v^2 A\{u(x)\} = 1 + \frac{1}{v^2} + A\left\{\int_0^x (x-t)\, u(t)dt\right\} \dots .(12)$$

Using convolution theorem of Aboodh transform on (12) and simplify, we have

$$A\{u(x)\} = \frac{1}{v^2 - 1} \dots \dots .(13)$$

Operating inverse Aboodh transform on both sides of(13), we have

$$u(x) = A^{-1}\left\{\frac{1}{v^2 - 1}\right\} = coshx \dots \dots (14)$$

which is the required exact solution of (11).

Application:3 Consider linear Volterra integro-differential equation of second kind

$$\left.\begin{array}{c} u'''(x) = -1 + \int_0^x u(t)dt \\ with\, u(0) = u'(0) = 1, u''(0) = -1 \end{array}\right\} \dots \dots (15)$$

Applying the Aboodh transform to both sides of (15) and using initial condition, we have

$$v^3 A\{u(x)\} = v + 1 - \frac{1}{v} - \frac{1}{v^2} + A\left\{\int_0^x u(t)\, dt\right\} \dots .(16)$$

Using convolution theorem of Aboodh transform on (16) and simplify, we have

$$A\{u(x)\} = \frac{1}{v^2 + 1} + \frac{1}{v(v^2 + 1)} \dots \dots .(17)$$

Operating inverse Aboodh transform on both sides of(17), we have

$$u(x) = A^{-1}\left\{\frac{1}{v^2+1}\right\} + A^{-1}\left\{\frac{1}{v(v^2+1)}\right\}$$
$$= cosx + sinx \ldots\ldots (18)$$

which is the required exact solution of (15).

Application:4 Consider linear Volterra integro-differential equation of second kind

$$\left.\begin{array}{c} u''(x) = x + \int_0^x (x-t)u(t)dt \\ with\, u(0) = 0, u'(0) = 1 \end{array}\right\} \ldots\ldots (19)$$

Applying the Aboodh transform to both sides of (19) and using initial condition, we have

$$v^2 A\{u(x)\} = \frac{1}{v} + \frac{1}{v^3} + A\left\{\int_0^x (x-t)\, u(t)dt\right\} \ldots (20)$$

Using convolution theorem of Aboodh transform on (20) and simplify, we have

$$A\{u(x)\} = \frac{1}{v(v^2-1)} \ldots\ldots. (21)$$

Operating inverse Aboodh transform on both sides of(21), we have

$$u(x) = A^{-1}\left\{\frac{1}{v(v^2-1)}\right\} = sinhx \ldots\ldots (22)$$

which is the required exact solution of (19).

11. CONCLUSION

In this paper, we have successfully developed the Aboodh transform for solving linear Volterra integro-differential equation of second kind. The given applications show that the exact solution have been obtained using very less computational work and spending a very little time. The proposed scheme can be applied for other linear Volterra integral equations and their system.

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